Synopsis of the seventh lecture in ECO 5315 (Risk Analysis)

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November 27, 2009

Introduction

The chapter on risk analysis addresses the impact of risk and uncertainty on how managers make decisions. There is a rich intellectual history behind Chapter 13 dating back to the middle of the 17th century when a French mathematician, physicist, and philosopher named Blaise Pascal proposed his famous "Wager". Pascal's wager presents faith in God as a choice made under uncertainty, and Pascal approached this problem through the lens of *expected value*.[1] Pascal's wager represents a seminal contribution, not only in philosophy and religion, but also in economics, as it is widely acknowledged to have brought about the advent of "decision theory", the heart of which is decision-making under risk and uncertainty.

In 1738, a Dutch-Swiss mathematician named Daniel Bernoulli published a paper entitled "Exposition of a New Theory on the Measurement of Risk" in which he proposed the concept of *expected utility* as a criterion for decision-making under risk and uncertainty. Bernoulli proposes the following coin-tossing game between Peter and Paul:

> "Peter tosses a coin and continues to do so until it should land "heads" when it comes to the ground. He agrees to give Paul one ducat if he gets "heads" on the very first throw, two ducats if he gets it on the second, four if on the third, eight if on the fourth, and so on, so that with each additional throw the number of ducats he must pay is doubled. Suppose we seek to determine the value of Paul's expectation."

A series of coin tosses is a classic example of a serially uncorrelated random variable[2]; just because heads or tails appears on the first toss, this does not in any way influence the outcome of subsequent coin tosses. The net effect of joining the implied probability distribution of Bernoulli's coin toss game with the payoffs described above is that the expected value of such a game is *infinite*. Therefore, it is impossible, using the expected value criterion, to value such a game. Bernoulli resolved this paradox (i.e., the impossibility of valuing payoffs where the expected value is infinite) by invoking the notion of *diminishing marginal utility*. Diminishing marginal utility implies that as a

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[1] Pascal's wager appears in note 233 of his Pensées.

[2] Serial correlation refers to the correlation of a variable with itself over successive time intervals. A serially uncorrelated sequence such as a series of coin tosses has the property that there is *no* correlation of the realized outcome of any given coin toss with outcomes of subsequent coin tosses.

person's wealth increases, there is a decline in the marginal utility which that person derives from each additional unit (dollar) of wealth. In other words, the incremental utility value of an additional $1,000 is higher for someone with a net worth of $1,000 compared with say an otherwise identical person with a net worth of $1,000,000. Bernoulli suggests the use of a logarithmic utility function; i.e., $U = \ln W$,[3] and he derives an equation which shows the relationship between the gambler's wealth and how much he should be willing to pay to play.[4]

The important takeaway from Bernoulli's thought experiment is that in order to determine the terms under which people are willing to take risks, we need information concerning the manner in which they trade risk and reward off against each other, and the tool that enables us to make this determination is the utility function. Remarkably, it was not until the publication in 1944 of John von Neumann's and Oskar Morgenstern's book entitled *Theory of Games and Economic Behavior* that Bernoulli's ideas were formalized in a systematic fashion and became part of the mainstream of economic thought.[5]

Expected Utility and Certainty Equivalent for a Risk Averter

By definition, anyone with a utility function characterized by diminishing marginal utility is *risk averse*; i.e., such an individual prefers a certain outcome over an uncertain outcome, other things equal. To see this, let's reconsider the numerical example from page 24 of the Chapter 13 lecture note, only here we'll assume that $U(W) = \sqrt{W}$. As before, initial wealth W_0 = $100, and winning or losing $20 represent equally probable events. Therefore, expected wealth is $E(W) = .5\ (80) + .5(120) = \$100 = W_0$ and expected utility is $E(U(W)) = .5\sqrt{80} + .5\sqrt{120} = 9.9494$. The certainty equivalent is the level of wealth that, if one had it with certainty, would provide the same level of utility as the risk itself. The certainty equivalent of wealth for a given risk is determined by setting the expected utility of the risk equal to the utility of the certainty equivalent of wealth; i.e., by setting $E(U(W))$ $= U(W_{CE})$ and solving for W_{CE}. In this case, since $U(W) = \sqrt{W}$, this implies that

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[3] Note that $U = \ln W$ has the property of diminishing marginal utility; marginal utility is always positive (i.e., $dU/dW = 1/W > 0$) but diminishing in wealth (i.e., $d^2U/dW^2 = -1/W^2 < 0$).

[4] Specifically, a millionaire should be willing to pay up to $10.94, a person with $1000 in initial wealth should be willing to pay up to $5.94, and a person with $2 in initial wealth should be willing to pay up to $2. See http://www.answers.com/topic/st-petersburg-paradox for more details on these calculations.

[5] As I noted in my "Synopsis of the sixth lecture in ECO 5315", this book is also famous for introducing the foundations for using game theory in economics.

$W_{CE} = 9.9494^2 = \$98.99$.[6] This implies that someone with initial wealth of \$100 and $U(W) = \sqrt{W}$ is *indifferent* between owning a lottery with equally likely payoffs of \$80 or \$120, compared with having \$98.99 for sure. However, owning \$100 for sure is better than owning \$98.99 for sure, since $U(100) = 10$ compared with $U(98.99) = 9.9494$. Consequently, in order to motivate this person to take the risk, it must either be made less risky and/or provide for a more generous expected payoff.[7]

Next, let's see what happens to the certainty equivalent of wealth if we reduce initial wealth from $W_0 = \$100$ to $W_0 = \$50$. Then $E(W) = .5(30) + .5(70) = \$50 = W_0$, expected utility is $E(U(W)) = .5\sqrt{30} + .5\sqrt{70} = 6.9219$, and certainty equivalent of wealth is $W_{CE} = 6.9219^2 = \$47.91$. The important takeaway here is that as one grows poorer, the *risk premium* grows larger (going from \$1.01 when $W_0 = \$100$ to \$2.09 when $W_0 = \$50$). As I noted in my blog entry entitled "Revisiting the Risky Investment Problem", this is because this utility function (as well as the other utility functions with which we have experimented) has the property of *decreasing (absolute) risk aversion*. Behaviorally, this implies that if initial wealth is lower (higher) you become more (less) risk averse. The reason this occurs is that as you grow poorer (wealthier), the utility consequence of a given risk increases (declines).

Expected Utility and Certainty Equivalent for a Risk Lover

In order to rationally explain the existence of economic institutions such as state lotteries and the gaming industry, we need to expand our theory a bit to allow for the possibility that people may exhibit (at least over some wealth levels) risk loving behavior.[8] This is shown on page 25 of the Chapter 13 lecture note. Here, we need a *convex* utility

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[6] Note that the *risk premium*, which represents the difference between expected wealth and its certainty equivalent, is smaller for the square root utility function that it was for the logarithmic utility function; here it is \$1.01, whereas in the case of the logarithmic utility function, it was \$2. Thus the square root utility function is a *less risk averse* utility function than the logarithmic utility function!

[7] For example, suppose the overall risk remains the same (i.e., the dispersion across the two states remains \$40, implying a standard deviation of \$20) but the bet has an expected value of \$1.01; i.e., suppose the loss is \$18.99 rather than \$20 and the gain is \$21.01 rather than \$20. Then $E(U(W)) = .5\sqrt{81.01} + .5\sqrt{121.01} = 10 = U(W_0)$.

[8] See Friedman, Milton and L. J. Savage, 1948, "The Utility Analysis of Choices Involving Risk", *The Journal of Political Economy*, Vol. 56, No. 4 (August), pp. 279-304, for a formal model that explains the joint existence of insurance and lotteries. However, it is generally assumed (and empirically well documented, I might add) in most of the business disciplines that managers, investors, consumers, etc. are risk averse.

function; i.e., a utility function that exhibits *increasing marginal utility*. Suppose $U(W) = W^2$, initial wealth W_0 = \$100, and we face the same risk as before; i.e., a 50% probability of winning \$20, and 50% probability of losing \$20.[9] Expected wealth is $E(W) = W_0$ = \$100, and expected utility is $E(U(W)) = .5(80^2) + .5(120^2) = 10,400$. In order to find the certainty equivalent of wealth, we set $E(U(W))$ equal to $U(W_{CE})$ and solve for W_{CE}. In this case, since $U(W) = W^2$, this implies that $W_{CE} = \sqrt{10,400} = \$101.98$. Note here that the risk premium is –\$1.98; consequently, a risk lover derives utility from bearing risk and is willing to pay for the opportunity to do so. For this risk lover, owning a lottery with equally likely payoffs of \$80 or \$120 is better than \$100 for sure. If you were a casino manager, you would love risk lovers because you can earn profit by offering unfair bets. For example, in the present case, this individual is indifferent between paying \$1.98 in order to take this risk compared with not gambling at all. If we factor this \$1.98 premium in, we find that the loss (net of the risk premium) is \$21.98 rather than \$20, and the gain (net of the risk premium) is \$18.02 rather than \$20. Then $E(U(W)) = .5(78.02^2) + .5(118.02^2) = 10,000 = U(100)$.

Expected Utility and Certainty Equivalent for a Risk Neutral Individual

The final case that we consider is risk neutrality. Someone who is risk neutral is *indifferent* about risk, which implies that the expected value of wealth is equal to the certainty equivalent of wealth. Risk neutral utility functions are linear in wealth, implying that marginal utility is *constant*. Suppose $U(W) = a + bW$; then $E(U(W)) = a + bE(W)$. Let $a = 0$ and $b = 1$; then $E(U(W)) = E(W)$, and $U(W_{CE}) = W_{CE}$. Using the same risk as in the previous section, this implies that $E(U(W)) = E(W) = W_{CE} = 100$, and the risk premium is $E(W) - W_{CE} = 0$.

Risk Preference Summary

- If $E(W) > W_{CE}$, then this individual is a *risk averter* because her risk premium ($E(W) - W_{CE}$) is positive;
- If $E(W) < W_{CE}$, then this individual is a *risk lover* because her risk premium ($E(W) - W_{CE}$) is negative; and
- If $E(W) = W_{CE}$, then this individual is *risk neutral* because her risk premium ($E(W) - W_{CE}$) is zero.

[9] Note that if $U(W) = W^2$, utility is always positive (i.e., $dU/dW = 2W > 0$) but *increasing* in wealth (i.e., $d^2U/dW^2 = 2 > 0$).