CHAPTER 13

Risk Analysis

Famous Risk and Uncertainty Quotes

"The evils of uncertainty must count for something".

Alfred Marshall (1920), in

Principles of Economics.

"Take calculated risks. That is quite different from being rash."

General George S. Patton

Game Plan

- Expected value model
- Expected utility model
- Risk preferences (risk aversion, risk neutrality, risk loving)
- Risk Analysis Class Problem

Making decisions under certainty

- Decisions are easier to make under certainty compared with uncertainty!
  - Under certainty, a decision can only lead to one outcome; the best decision involves making a choice that is consistent with one's preferences.
  - e.g., suppose you prefer more money to less and you are faced with two investment choices that cost $50 each; Investment A pays off $100, whereas Investment B pays off $120.
  - Given your preferences, the logical choice is to invest in B, since B nets $70 whereas A only nets $50.

Making decisions with risk

- In a world where there is risk, decisions are more complicated!
  - With risk, a decision can result in more than one outcome.
  - e.g., suppose Investment A has a 50% chance of paying off $50, and a 50% chance of paying off $150, whereas Investment B has a 50% chance of paying off $0 and a 50% chance of paying off $240. Which investment do you prefer now?
    - Investment A has an *expected* payoff of .5($50) + .5($150) = $100, whereas Investment B has an *expected* payoff of .5($0) + .5($240) = $120. Do you still prefer A over B? Why or why not?

Definitions: States of nature and probabilities

- An uncertain outcome means that more than 1 possible outcome may occur.
  - "States" of nature – rainy vs. sunny, market up vs. market down, recession vs. expansion, heads vs. tails, etc.
  - Each possible "state" or outcome has a probability associated with it.
  - Probabilities sum to one.

Definitions: Expected Value

- Suppose there are n states of the world for a random variable that we'll call x.
- The state-contingent value for x is given as x_s.
- The probability that state s will occur is given by p_s.
- The sum of all n values for p_s is equal to 1.0; i.e., $$\sum_{s=1}^{n} p_s = 1.0.$$
- The *expected value* of x is its probability-weighted average value; i.e., $E(x) = \sum_{s=1}^{n} p_s x_s$.

Definitions: Variance and standard deviation

- The variance σ_x^2 is calculated as follows:

$$\sigma_x^2 = E[(x - E(x))^2] = \sum_{s=1}^{n} p_s (x_s - E(x))^2.$$

- The standard deviation σ_x measures the spread, or dispersion around the expected value; it is calculated by taking the square root of the variance.
- Standard deviation is commonly used (and often misused) as a measure of risk.

The Expected Value Decision Rule

	WIN	*LOSE*	*Expected Value*
A	$10 prob 0.5	$10 prob 0.5	$0
B	$20 prob 0.5	$20 prob 0.5	$0
C	$14 prob 0.5	$13 prob 0.5	$0.5
D	$0 prob 0.5	$0 prob 0.5	$0
E	$17 prob 0.5	$14 prob 0.5	$1.5
F	$20 prob 0.55	$20 prob 0.45	$2

- Many (though not all) people will prefer C to A or B. Similarly, many prefer F to A or B. In both cases, these choices are consistent with maximizing expected value.
- However, many (though, again not all) people will prefer D to C, E or F; such choices *do not* maximize expected value.

The Expected Value Decision Rule

	WIN	*LOSE*	*EV (Std. Dev.)*
A	\$10 prob 0.5	\$10 prob 0.5	\$0 (10)
B	\$20 prob 0.5	\$20 prob 0.5	\$0 (20)
C	\$14 prob 0.5	\$13 prob 0.5	\$0.5 (13.50)
D	\$0 prob 0.5	\$0 prob 0.5	\$0 (0)
E	\$17 prob 0.5	\$14 prob 0.5	\$1.5 (15.50)
F	\$20 prob 0.55	\$20 prob 0.45	\$2 (19.90)

- The standard deviations have been added to the previous table.
- People typically prefer higher expected value and lower risk, *cet. par.*
  - Those who are MORE RISK AVERSE (MRA) will tend to sacrifice high expected value to obtain lower risk – thus D (not gambling) is the most attractive to the MRA types.
  - Those who are LESS RISK AVERSE (LRA) will pay less attention to risk focusing mainly on EV. So C, E and F may be attractive for the LRA types.

The St. Petersburg Paradox

- Daniel Bernoulli (1738) proposes the following gamble:
  - "Peter tosses a coin and continues to do so until it should land "heads" when it comes to the ground. He agrees to give Paul one ducat if he gets "heads" on the very first throw, two ducats if he gets it on the second, four if on the third, eight if on the fourth, and so on, so that with each additional throw the number of ducats he must pay is doubled. Suppose we seek to determine the value of Paul's expectation."
  - Without any loss of generality, let's change the monetary unit from the ducat to the U. S. dollar.

The St. Petersburg Paradox

- For starters, note that the total probability of an infinite series of fair coin tosses sums to 1 (i.e., $\sum_{i=1}^{\infty} .5^i = 1$), so Professor Bernoulli has proposed a valid probability distribution for this gamble.

- Next, we calculated the expected value of this gamble.
  - If the coin comes up heads on the first toss (when $i = 1$), then the payoff is $\$2^{i-1} = \$2^0 = \$1$, and the game ends. However, if the coin comes up tails on the first toss, it is tossed again.
  - If the coin comes up heads on the second toss (when $i = 2$), the payoff is $\$2^{i-1} = \$2^1 = \$2$, and the game ends. However, if the coin comes up tails on the second toss, it is tossed again, and so forth.

The St. Petersburg Paradox

- The probability that it will take *n* coin tosses in order for heads to come up is $.5^n$, and the payoff after the n^{th} coin toss is $\$2^{n-1}$; thus the expected value of this game is

$$EV = \sum_{i=1}^{\infty} .5^i 2^{i-1} = \sum_{i=1}^{\infty} .5 \Rightarrow \infty .$$

- Bernoulli goes on to note, "My ... cousin discussed this problem in a letter to me asking for my opinion. Although the standard calculation shows that the value of Paul's expectation is infinitely great, it has, he said, to be admitted that any fairly reasonable man would sell his chance, with great pleasure, for twenty ducats (dollars). The accepted method of calculation does, indeed, value Paul's prospects at infinity though no one would be willing to purchase it at a moderately high price."

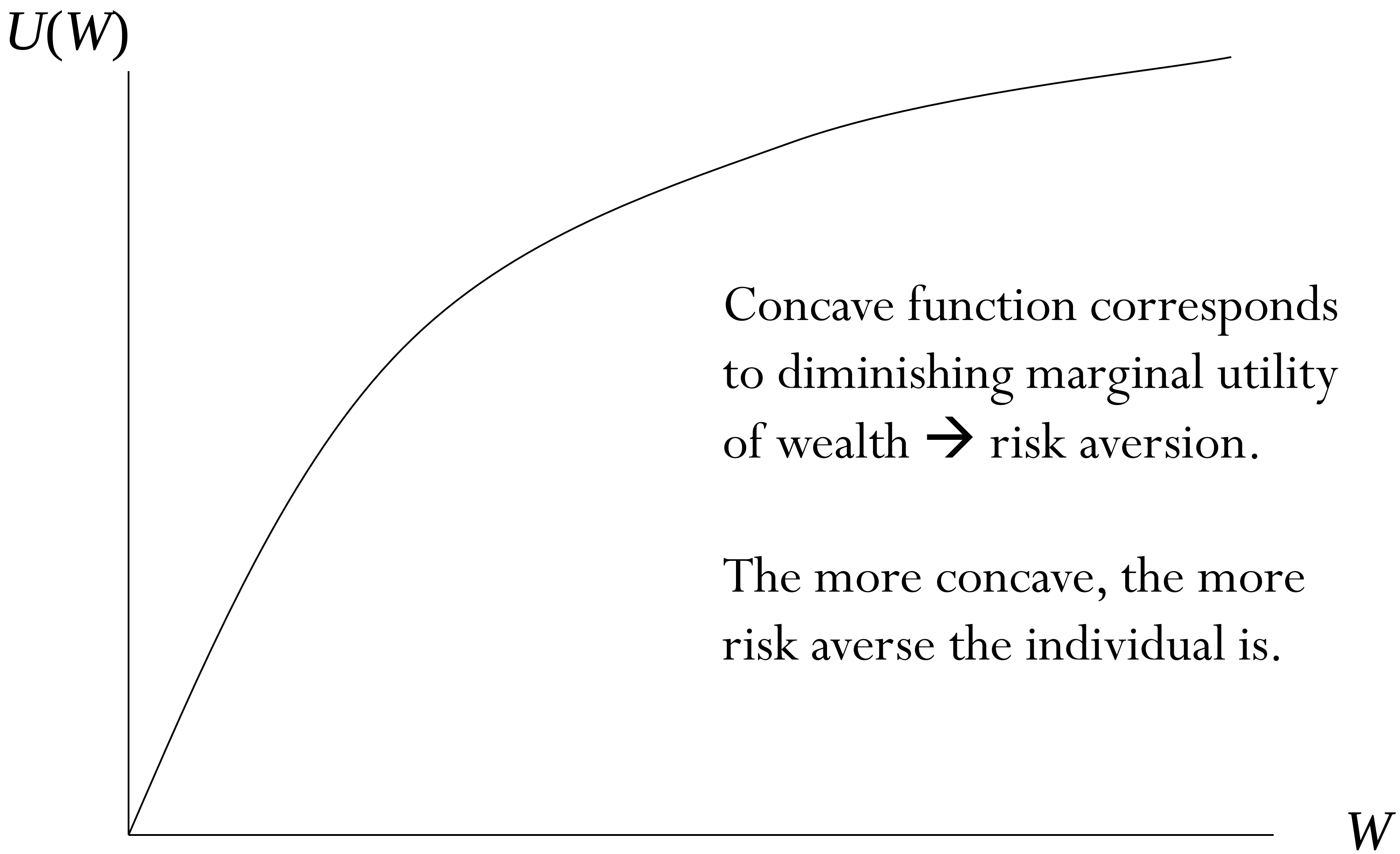
Risk Aversion and Expected Utility

- Most individuals behave (most of the time) *as if* they are *risk averse*.
- Risk averse consumers prefer a certain outcome over an uncertain outcome with the same expected value.
  - If two outcomes with the same expected value are both risky, risk averse consumers prefer the outcome with less risk.
  - If two outcomes are both equally risky, risk averse consumers prefer the one with higher expected value.
- This risk/reward tradeoff is captured by the consumer's *utility function*.

Risk Aversion & Diminishing Marginal Utility

- Risk averse utility functions have the property of *diminishing marginal utility*.
  - Diminishing marginal utility implies that as a person's wealth increases, there is a decline in the marginal utility that person derives from each additional unit (dollar) of wealth.
  - In other words, the utility value of an additional $1,000 is much higher for someone with a net worth of $1,000 compared with an otherwise identical person with a net worth of $1,000,000.

Graphical Depiction of Risk Aversion



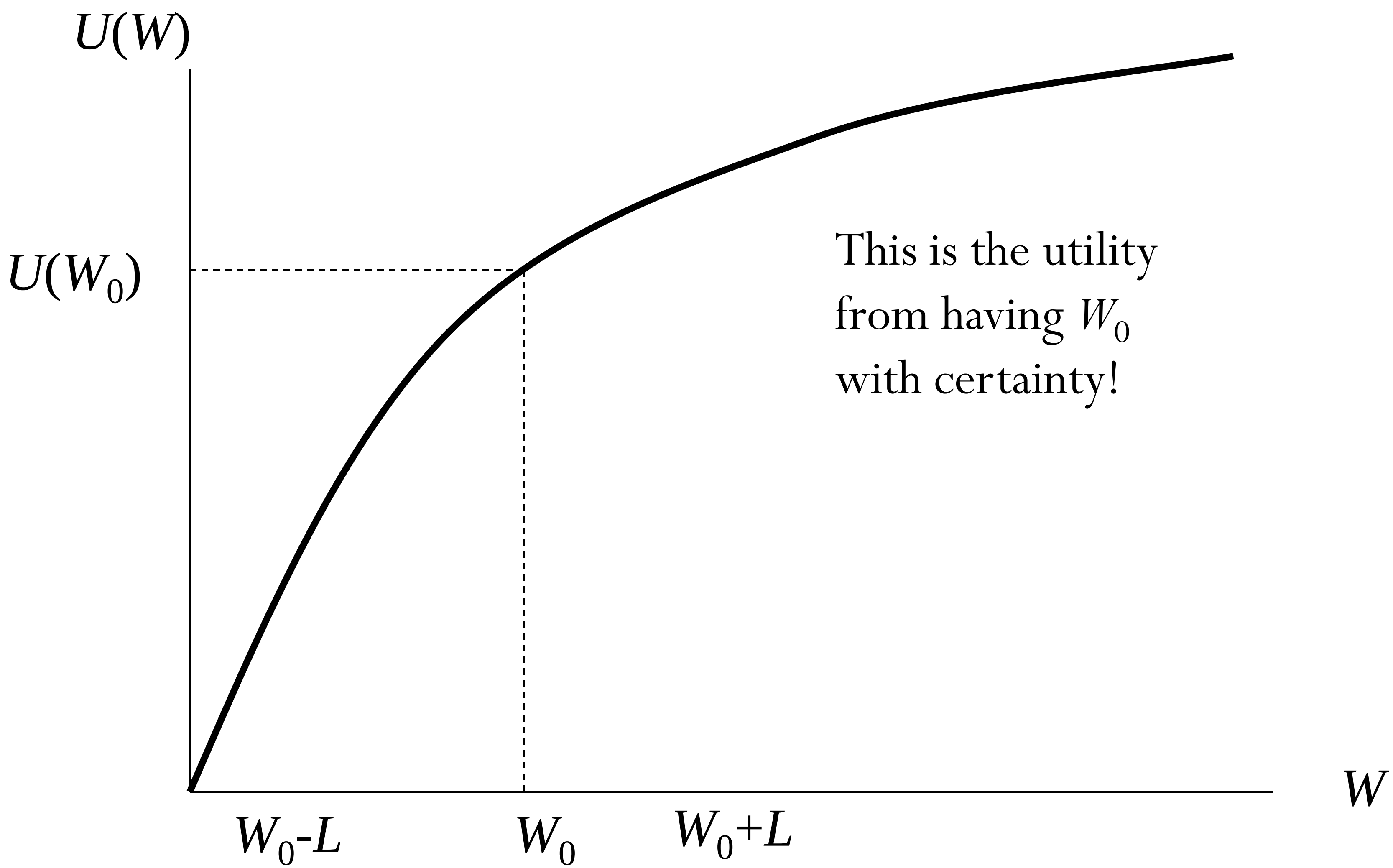
Concave function corresponds to diminishing marginal utility of wealth → risk aversion.

The more concave, the more risk averse the individual is.

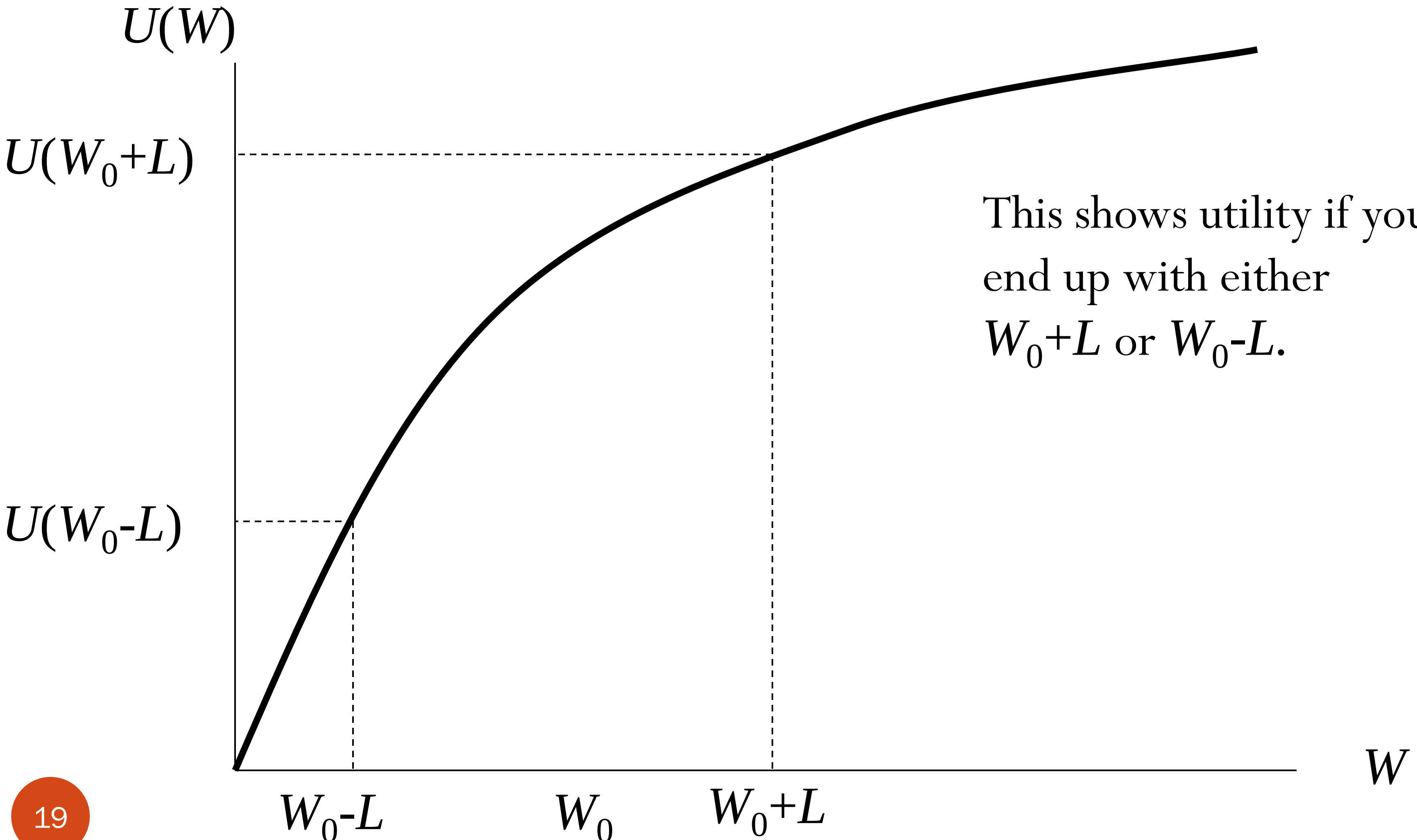
Expected Utility Example

- Suppose your utility function is *logarithmic*; i.e., $U(W) = \ln(W)$ and you start with wealth W_0.
- There is a 50% chance of losing $\$L$ and 50% chance of winning $\$L$ (i.e., this is an "actuarially fair" bet).
- $E(W) = (.5)(W_0+L)+(.5)(W_0-L) = W_0$.
- Is expected utility *with* the gamble higher or lower than expected utility without the gamble?
- Is $E(U(W)) = (.5)\ln(W_0+L)+(.5)\ln(W_0-L) >, <,$ or $= \ln(W_0)$?
- Note that $U(W)$ is a risk averse utility function!

Utility from W_0 with Certainty

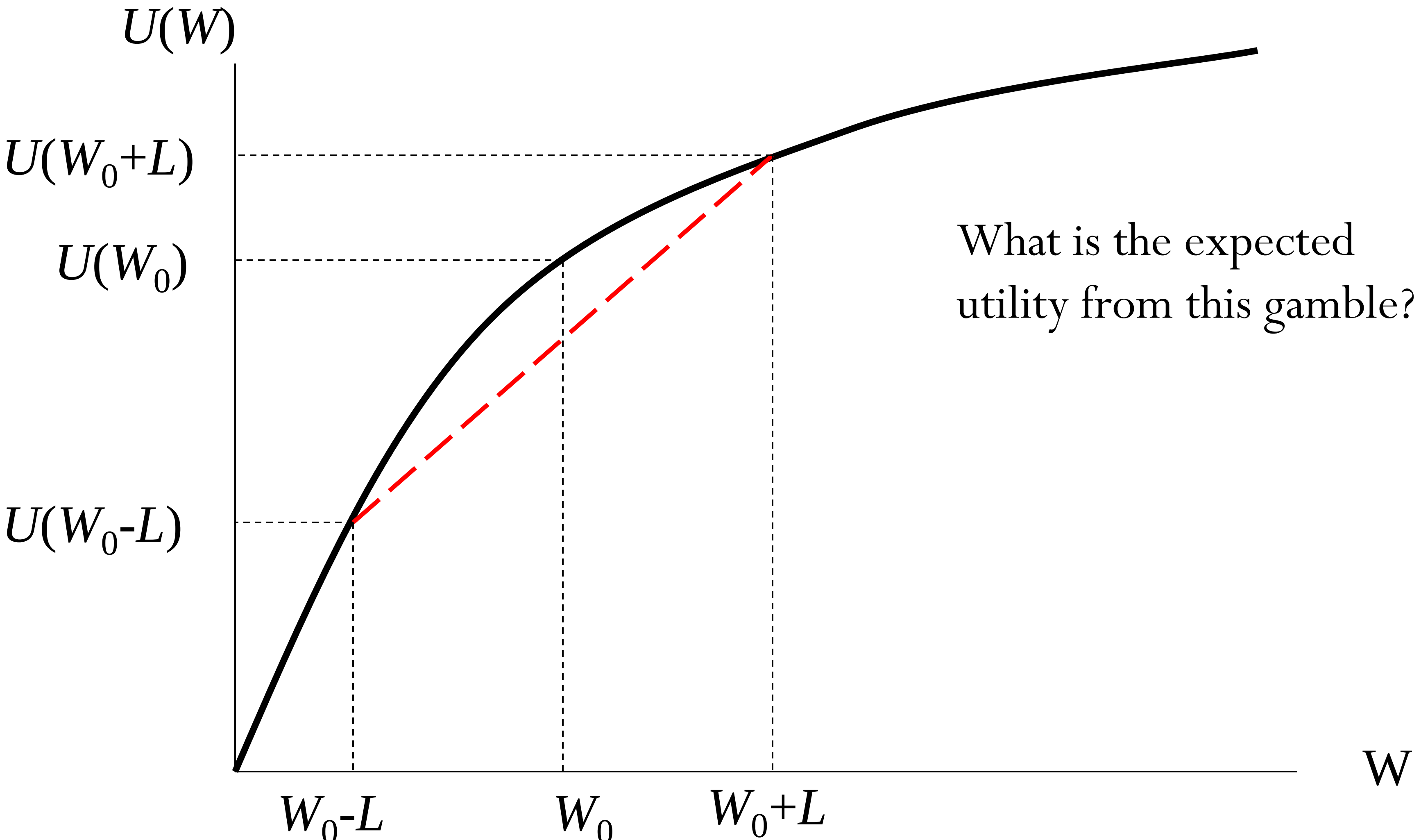


Utility from W_0+L or W_0-L



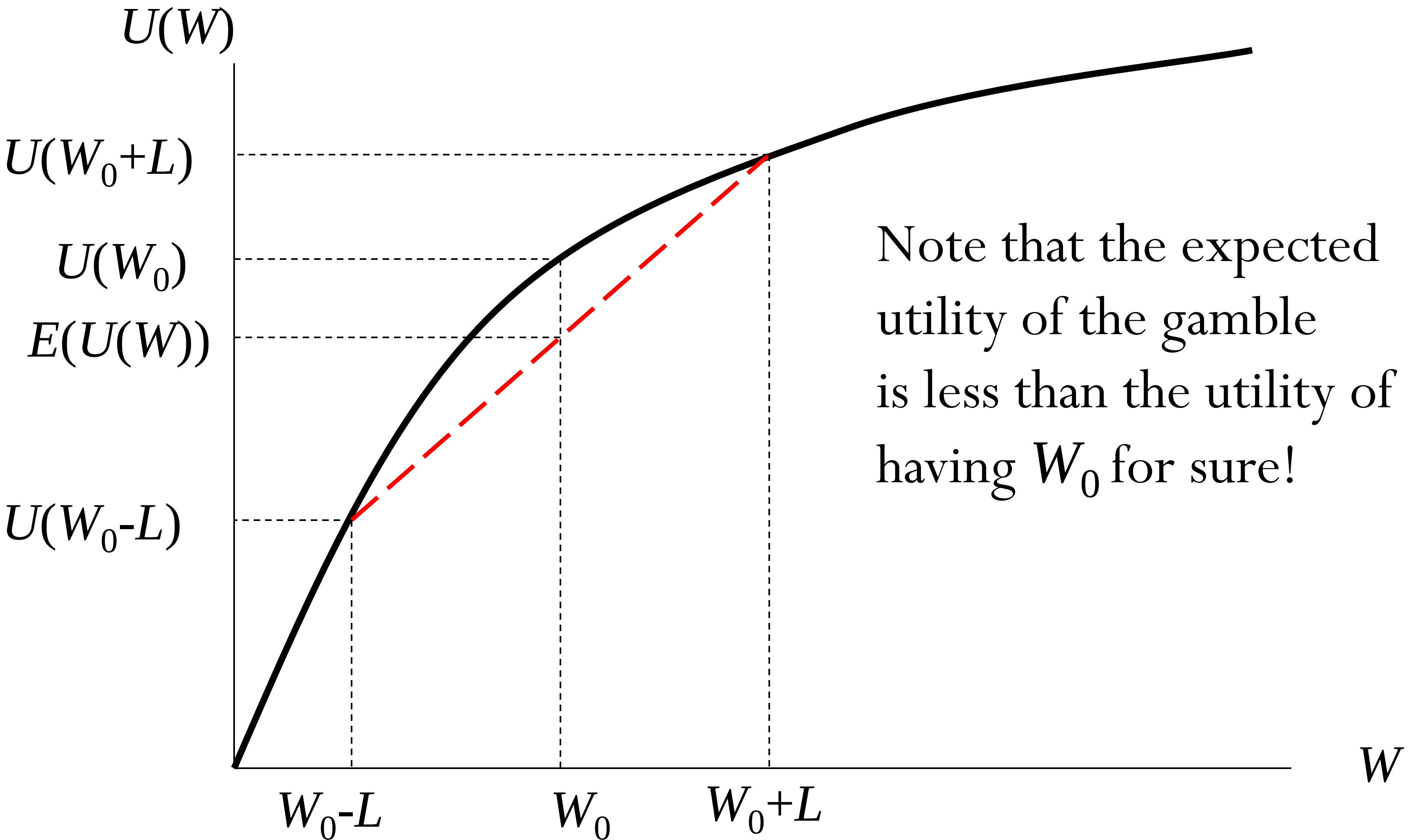
This shows utility if you end up with either W_0+L or W_0-L.

Utility comparisons



What is the expected utility from this gamble?

Expected Utility from Gamble

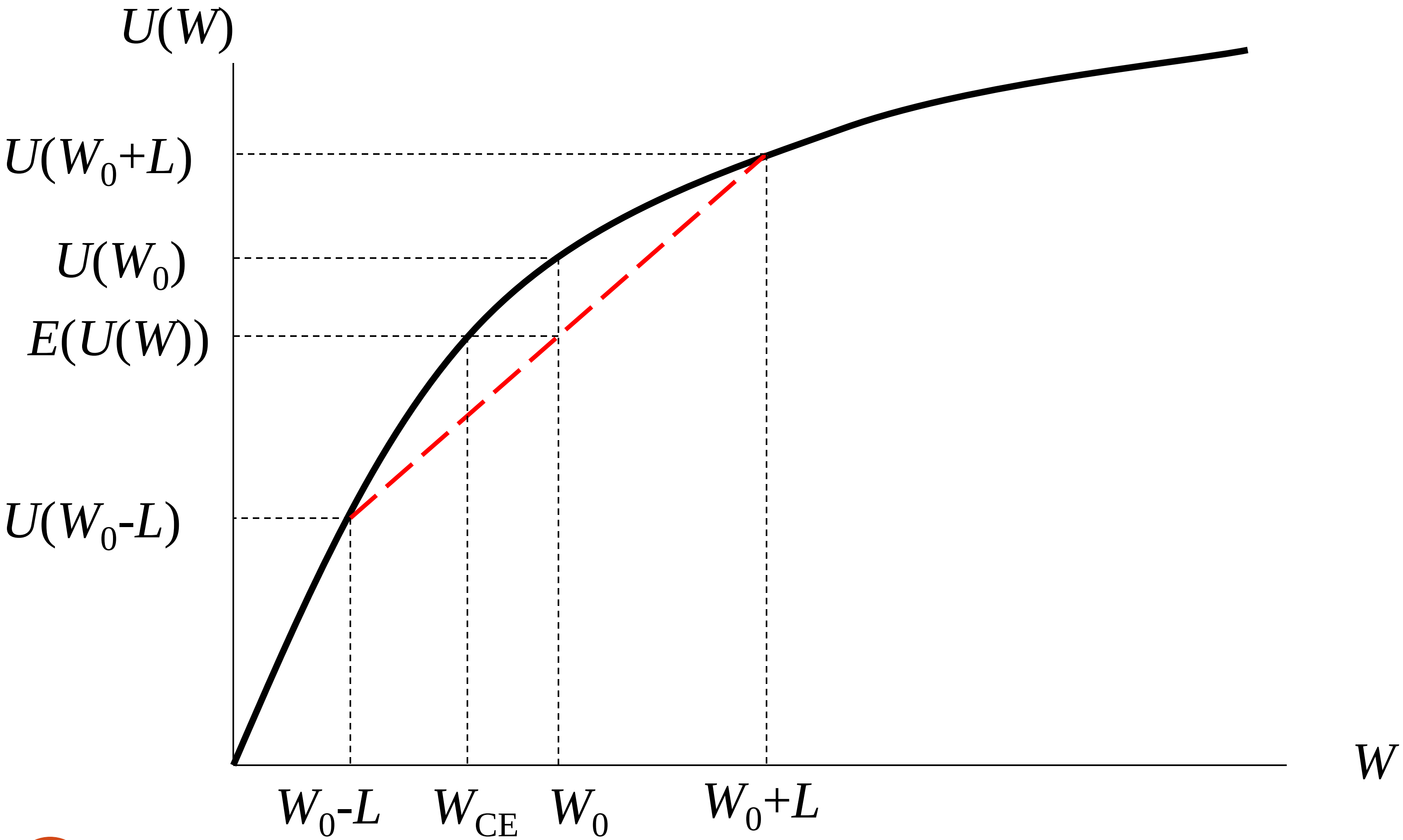


Note that the expected utility of the gamble is less than the utility of having W_0 for sure!

Certainty Equivalent Wealth

- We know that expected utility from gamble is *less* than utility of having W_0 for sure.
- "Certainty Equivalent Wealth" is the level of wealth that, if we had it with certainty, would provide us with the same level of expected utility as the gamble itself!

Certainty Equivalent Wealth



Numerical Example of W_{CE}

- $U(W) = \ln(W)$
- You have initial wealth $W_0 = \$100$, but face a 50% probability of winning \$20, and 50% probability of losing \$20.
- Therefore, $E(U(W)) = .5\ln(80) + .5\ln(120) = 4.585$.
- What amount of certain wealth gives you utility of 4.585?
  - Set $E(U(W)) = U(W_{\text{CE}})) \Rightarrow 4.585 = \ln(W_{\text{CE}})$.
  - Since $W_{\text{CE}} = e^{\ln(W_{CE})}$, this implies that $W_{\text{CE}} = e^{4.585} = \$98$.
- Therefore, this person is indifferent between having a 50/50 chance of \$80 or \$120, and having \$98 with certainty!

Risk Loving Behavior

- In the previous set of slides, we showed (among other things) that risk aversion implies that utility is concave; i.e., that $E(U(W)) < U(E(W))$.
- What if utility is convex? Suppose $U = W^2$; then $E(U(W)) > U(E(W))$.
- As before, initial wealth $W_0 = \$100$, there's a 50% probability of winning \$20, and 50% probability of losing \$20.
- Therefore, expected utility $= .5(80^2)+.5(120^2) = 10{,}400$.
- What amount of certain wealth gives you utility of 10,400? Set $W_{CE}^2 = 10{,}400$; therefore, $W_{CE} = \$101.98$.
- This person would be indifferent between 50/50 chance of \$80 or \$120, and having \$101.98 with certainty!

Risk Neutral Behavior

- What if utility is linear? Suppose $U = W$; then $E(U(W)) = U(E(W))$.
- As before, initial wealth $W_0 = \$100$, there's a 50% probability of winning \$20, and 50% probability of losing \$20.
- Expected utility = $.5(80)+.5(120) = 100$.
- What amount of certain wealth gives you utility of 100? Since $U(W_{CE}) = W_{CE} = \$100$, we have our answer!
- This person is indifferent between having a (risky) expected value of \$100 and \$100 with certainty!

Revisiting the Risky Investment Problem!

- On page 5 of this lecture note, I proposed the following problem:

  Suppose Investment A has a 50% chance of paying off \$50, and a 50% chance of paying off \$150, whereas Investment B has a 50% chance of paying off \$0 and a 50% chance of paying off \$240. Which investment do you prefer now?

  - Initially answer this question by assuming that $W_0 = \$100$ and $U(W) = W^{0.5}$.
  - Then redo this problem, assuming that $W_0 = \$200$.

Risk Analysis Class Problem

- Blanche has identified a risky venture opportunity. This venture will have one of two possible outcomes: It will be a flop, or it will be successful. If it is a flop, it will make \$0. If it is successful, it will make \$120,000. There is an 80% chance it will be a flop, and a 20% chance that it will be successful. It does not cost Blanche anything to undertake this venture. Blanche has initial wealth of \$40,000, and her utility function is $U(W) = W^{0.5}$, where W is her wealth.
  A. What is Blanche's certainty equivalent for this risky venture opportunity?

Risk Analysis Class Problem

- Now Blanche hits upon another idea. She finds 10 members of her EMBA cohort, and decides to try to "sell" the opportunity to invest in this venture for $2,000 per investor. Each of the 10 cohort members that Blanche approaches with this investment offer has an initial wealth of $90,000, and a utility function $U(W) = W^{0.5}$, where W is the cohort member's wealth.
- If a cohort member invests $2,000 with Blanche, then:
  - If the venture is successful, Blanche keeps the investor's $2,000 and pays the cohort member 1/10th of the upside (that is, $12,000). This means the investor enjoys a net gain of $10,000, and a final wealth level of $100,000.
  - If the venture is a flop, however, then Blanche will keep the investor's $2,000 and pay the investor nothing in return. The investor suffers a net loss of $2,000, and a final wealth of $88,000.

Risk Analysis Class Problem

B. I-Banker Joe is one of the ten cohort members Blanche approaches. What is Joe's certainty equivalent for the deal that Blanche is offering? If Joe's only two options are either to accept Blanche's offer, or to maintain his initial wealth of $90,000 for sure, will Joe accept Blanche's offer?

C. Is Blanche better off selling-off her investment opportunity to 10 cohort members (under the terms described above), or is she better off taking the investment opportunity herself?