Synopsis of the third lecture in ECO 5315 (Production Theory)

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Introduction

Production theory provides insights concerning the most cost effective way to employ the firm's resources to produce output. In the vernacular, this is all about getting the most "bang for the buck". In order to keep the analysis as simple as possible, we focus our attention on how two inputs (capital (K) and labor (L)) can be used in order to produce an output (Q). The *production function* relates the quantity of output (Q) as a function of the inputs; specifically, this implies that $Q = f(L, K)$.

Law of Diminishing Returns

A ubiquitous principle which underlies production theory, ECO 5315, and life in general is that tradeoffs are pervasive. The way that this principle is reflected in production theory is in the form of the so-called "law of diminishing returns". [1] The application of this law to production theory implies that when managers add equal increments of one input while holding other input levels constant, the incremental increases in output eventually diminish.

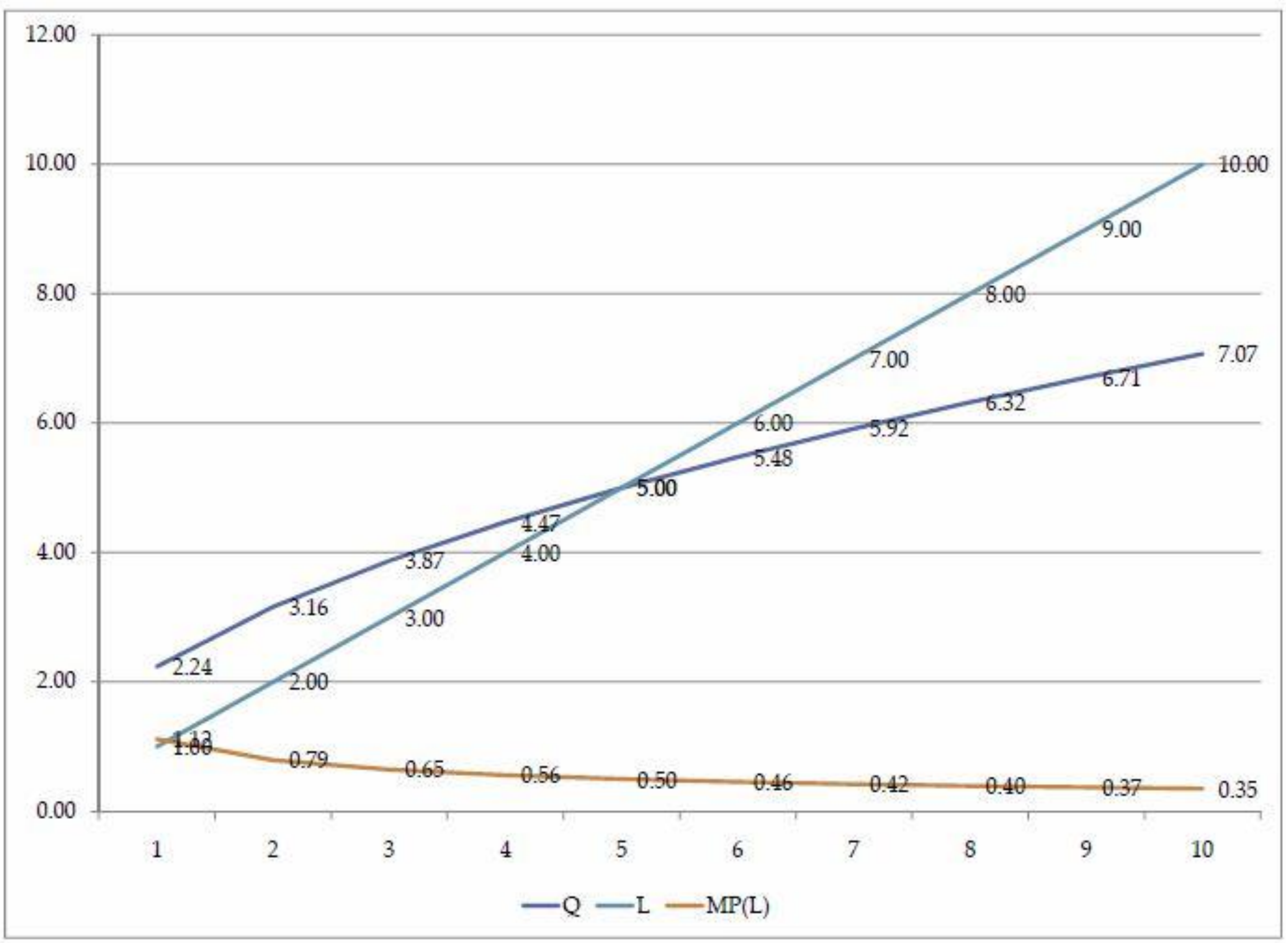
To illustrate this law, let's look at how it applies to a two-input production function of the form $Q = f(L, K)$. Consider a simple version of the Cobb-Douglas production function, $Q = L^{.5}K^{.5}$.[2] The marginal product of labor (MP_L) indicates the rate at which output increases as we add an additional unit of labor, while holding the level of capital constant. Mathematically, $MP_L = \partial Q / \partial L = .5L^{-.5}K^{.5} > 0$; MP_L *cannot* be negative since labor and capital inputs are typically positive. The law of diminishing returns implies that if we increase L at a constant rate while holding K constant, then Q increases at a *decreasing* rate; i.e., the marginal product of labor *declines* as more of the labor input is added. [3]

Next, let's apply our math to a simple numerical example. Specifically, let's fix the number of capital inputs at $K = 5$, and allow L to vary from 1 unit to 10 units. For each value of L, we'll calculate Q and MP_L, and also graph our results:

Table 3.1. Units of Output (Q) and MP_L when $K = 5$ units and L varies from 1 to 10.

Q	L	MP_L
2.24	1.00	1.12
3.16	2.00	0.79
3.87	3.00	0.65
4.47	4.00	0.56
5.00	5.00	0.50
5.48	6.00	0.46
5.92	7.00	0.42
6.32	8.00	0.40
6.71	9.00	0.37
7.07	10.00	0.35

Figure 3.1. Units of Output (Q) and MP_L when $K = 5$ units and L varies from 1 to 10.



This numerical example clearly illustrates the law of diminishing returns as it pertains to production theory. Specifically, when we employ 5 units of capital and vary the number of labor inputs from 1 to 10, then output increases at a *decreasing* rate, starting at Q = 2.24 units when K = 5 and L = 1, and ending at Q = 7.07 units when K = 5 and L = 10. Output increases at a decreasing rate *solely* because the marginal product of labor *decreases* at a decreasing rate. With every unit increase in L, MP_L declines at a decreasing rate.[4]

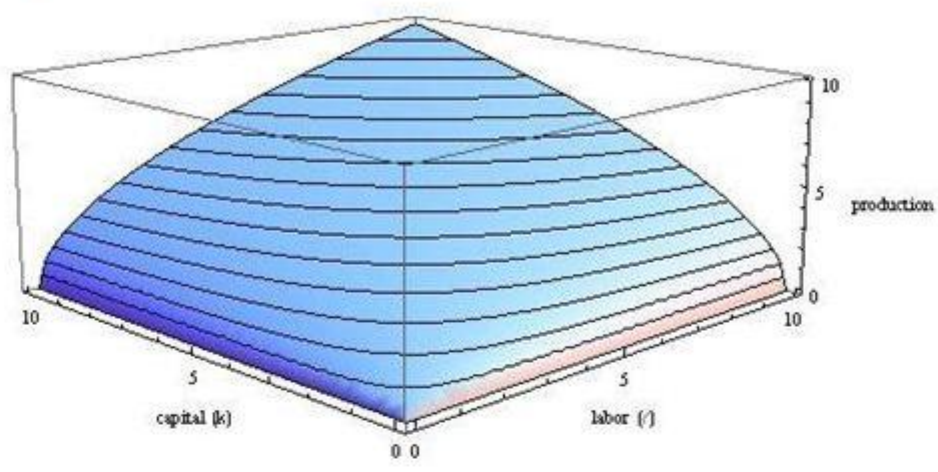
Returns to Scale

Given that the law of diminishing returns applies to all of our inputs into the production process, we are interested in finding the "optimal" mix of these inputs that maximizes output for a given cost. What's optimal depends upon the extent to which the firm enjoys *returns to scale*. Increasing returns to scale implies that doubling the firm's inputs more than doubles its output. Decreasing returns to scale implies that output will increase by *less* than 100% if the firm's inputs are doubled. Finally, constant returns to scale implies that output will double if the firm's inputs are doubled.

Whether the firm has increasing, constant, or decreasing returns to scale depends upon the *output elasticities* of its inputs. Output elasticity for a given measures the percentage change in output associated with a percentage change in a given input. Consequently, the output elasticity of labor (η_L) is $\eta_L = (L/Q)(\partial Q/\partial L)$ and the output elasticity of capital (η_K) is $\eta_K = (K/Q)(\partial Q/\partial K)$. If the sum of the output elasticity of labor plus the output elasticity of capital exceeds 1; i.e., if $\eta_L + \eta_K > 1$, then returns to scale are *increasing*. However, if $\eta_L + \eta_K < 1$, then returns to scale are *decreasing*, and if $\eta_L + \eta_K = 1$, then returns to scale are *constant*.

Let's revisit our old friend $Q = L^{.5}K^{.5}$. Does this production function exhibit increasing, constant, or decreasing returns to scale? The answer can be found by calculating the values for η_L and η_K. However, before we do any math, first look at its picture in Figure 3.2.

Figure 3.2. Picture of $Q = L^{.5}K^{.5}$, where L and K vary from 0 to 10.

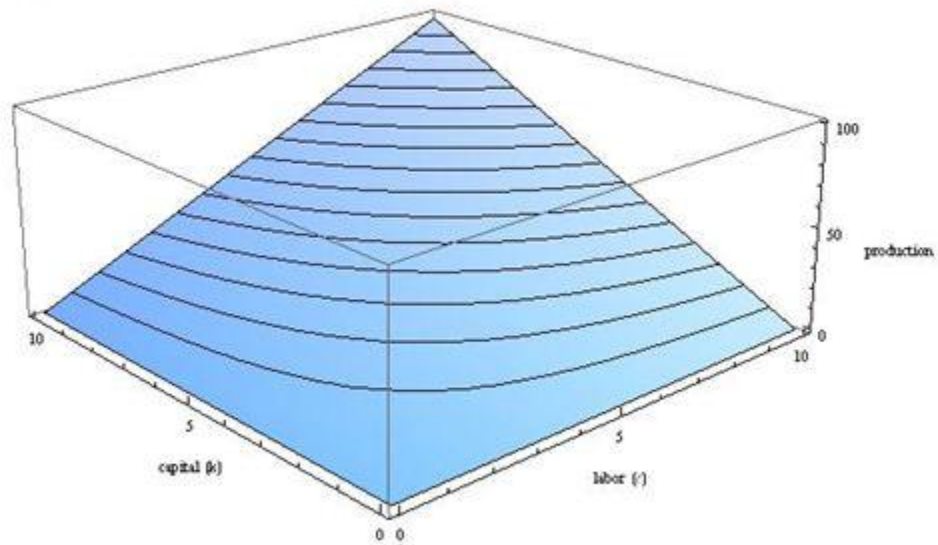


In Figure 3.2, Q is calculated for all possible combinations of L and K, where values for these inputs are allowed to vary from 0 to 10. Since $Q = L^{.5}K^{.5}$, when $L = K = 0$, then

$Q = (0 \times 0)^{.5} = 0$, and when $L = K = 10$, then $Q = (10 \times 10)^{.5} = 10$. Note that $Q = (5 \times 5)^{.5} = 5$ when $L = K = 5$; this implies that this production function exhibits constant returns to scale, since a doubling of the number of labor inputs (from 5 to 10) doubles the output. Furthermore, the math confirms this, since the output elasticity of labor $\eta_L = 0.5$, and the output elasticity of capital $\eta_K = 0.5$, so $\eta_L + \eta_K = 0.5 + 0.5 = 1$.[5]

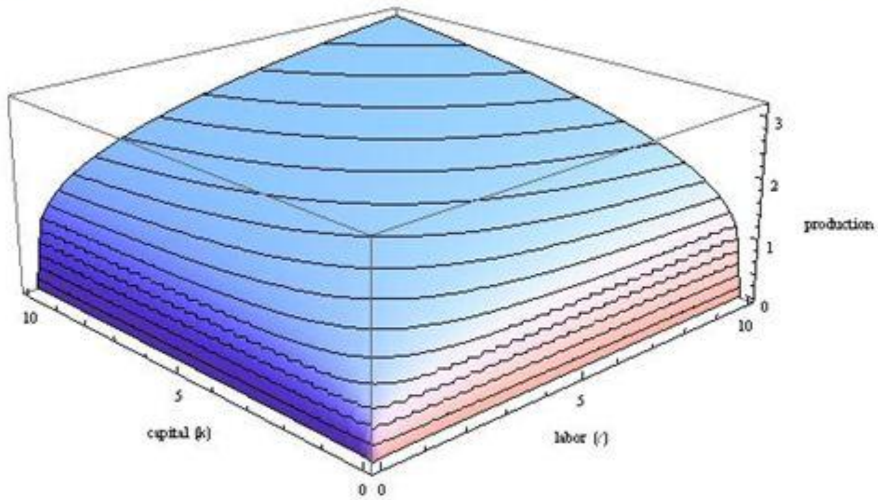
Next, let's consider a production function that has increasing returns to scale. To make it as simple as possible, let's consider the function $Q = LK$. We obtained this production function by doubling the values for each of the exponents associated with the labor and capital inputs in the constant returns to scale production function. Consequently, the overall output elasticity of this production function is $\eta_L + \eta_K = 1 + 1 = 2$. In Figure 3.3, Q is calculated for all possible combinations of L and K, where values for these inputs are allowed to vary from 0 to 10. Since $Q = LK$, when $L = K = 0$, then $Q = 0$, and when $L = K = 10$, then $Q = 10 \times 10 = 100$. Note that $Q = 5 \times 5 = 25$ when $L = K = 5$; this implies that this production function exhibits increasing returns to scale, since in this case a doubling of the number of labor inputs (from 5 to 10) *quadruples* the output.

Figure 3.3. Picture of $Q = LK$, where L and K vary from 0 to 10.



Finally, let's consider a production function that has decreasing returns to scale. Let's consider the function $Q = L^{.25}K^{.25}$. We obtained this production function by halving the values for each of the exponents associated with the labor and capital inputs in the constant returns to scale production function. Consequently, the overall output elasticity of this production function is $\eta_L + \eta_K = 0.25 + 0.25 = 0.5$. In Figure 3.4, Q is calculated for all possible combinations of L and K, where values for these inputs are allowed to vary from 0 to 10. Since $Q = L^{.25}K^{.25}$, when $L = K = 0$, then $Q = 0$, and when $L = K = 10$, then $Q = 10^{.25} \times 10^{.25} = 3.16$. Note that $Q = 5^{.25} \times 5^{.25} = 2.24$ when $L = K = 5$; this implies that this production function exhibits decreasing returns to scale, since in this case a doubling of the number of labor inputs (from 5 to 10) increases output by only 41%.

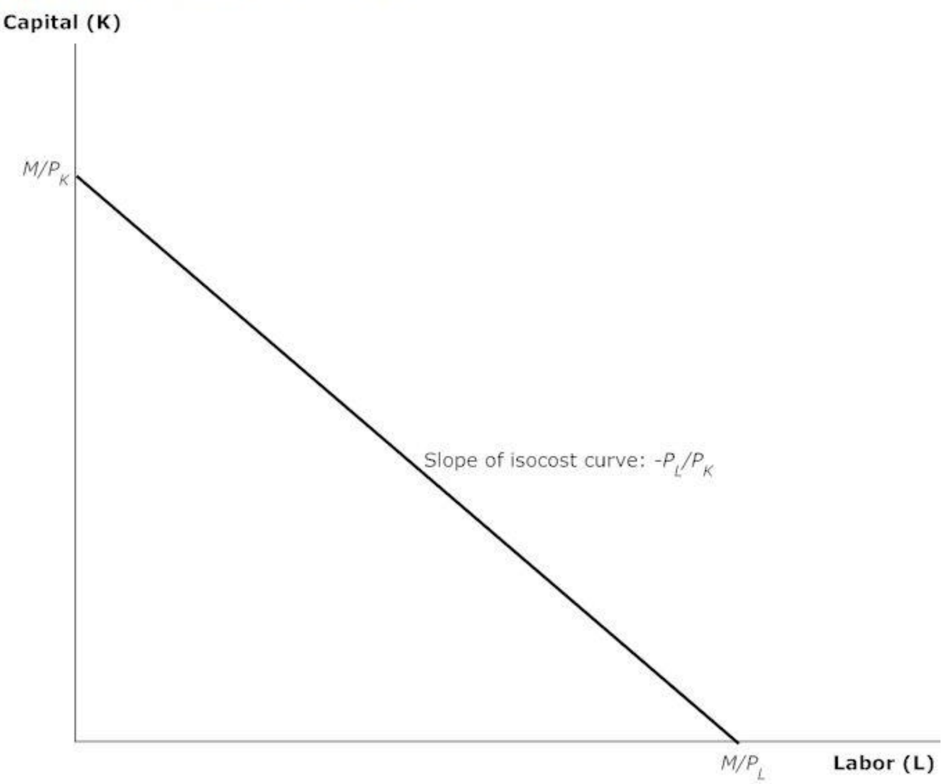
Figure 3.4. Picture of $Q = L^{.25}K^{.25}$, where L and K vary from 0 to 10.



The Bottom Line

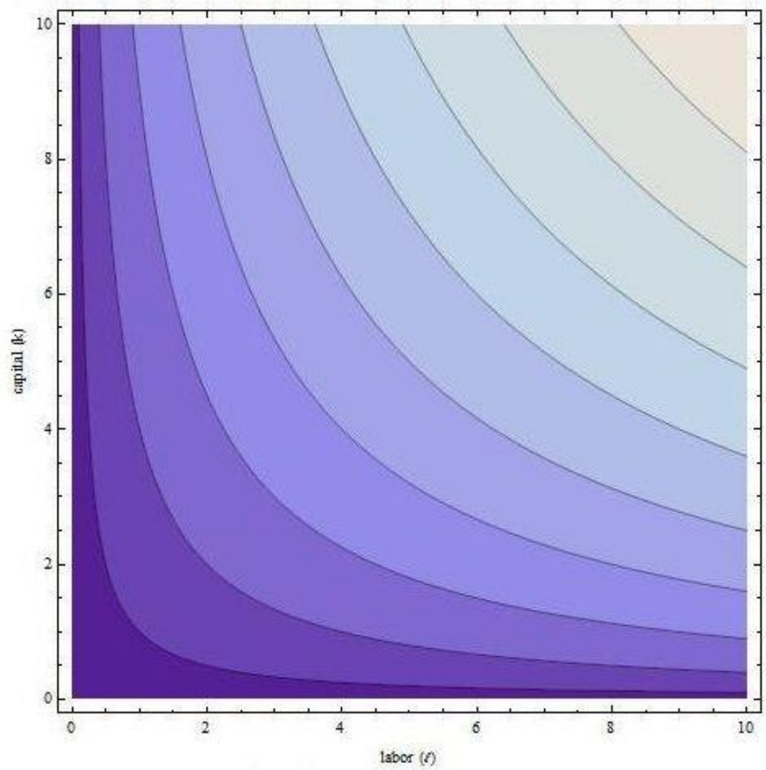
We conclude this synopsis by actually showing *how* to determine the most cost effective way to employ the firm's resources in a world characterized by the law of diminishing returns and returns to scale. Suppose your firm produces one product by employing L units of labor and K units of capital. The market prices for one unit of capital and one unit of labor are P_K and P_L, and you have budgeted M dollars to spend on capital and labor. Then your budget

Figure 3.5. Picture of an isocost line.



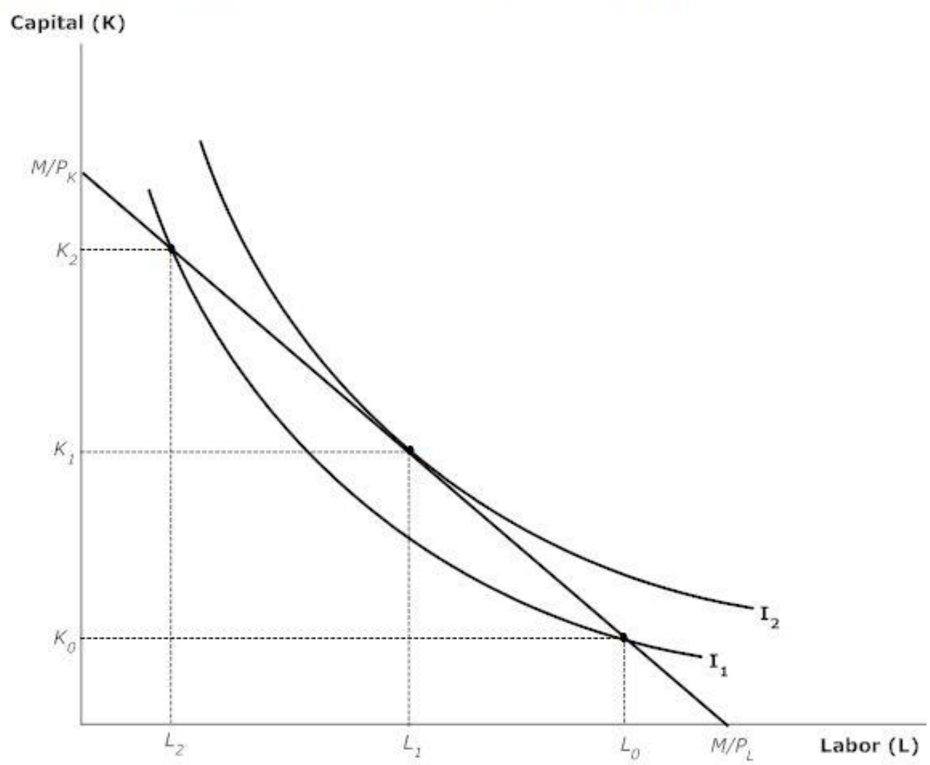
constraint is given by equation (4.4) in the textbook; i.e., $P_L L + P_K K = M$. Solving this equation for K yields equation (4.5) in the textbook; i.e., $K = \frac{M}{P_K} - \frac{P_L}{P_K} L$. This *isocost* line appears in Figure 3.5.[6]

Figure 3.6. Picture of isoquants for the production function $Q = L^{.5}K^{.5}$.



Now that we know how much our inputs are going to cost, the next step is to figure out how many inputs of L and K to employ. Irrespective of whether we have increasing, constant, or decreasing returns to scale, we can take any of the 3 dimensional pictures shown in Figures 3.2, 3.3, or 3.4 and collapse them into two dimensions. For example, Figure 3.6 provides a 2 dimensional version of Figure 3.2. The curves shown in Figure 3.6 are called *isoquants* (which literally means "same quantity"), and each isoquant comprises all points having the same height in the production surface shown in Figure 3.2. The isoquants near the lower left corner of Figure 3.6 consist of input bundles that produce less output than the input bundles represented by the isoquants near the upper right corner.

Figure 3.7. Minimization of Cost for a Given Output.



The final step involves incorporating the isocost line and isoquants into the same graph and using this information to infer which input bundle maximizes output for a given cost. Figure

3.7 shows how this works. In this figure, the *optimal* input bundle is represented by the (K_1, L_1) point. How do we know that (K_1, L_1) is optimal? Note that the input bundles represented by (K_2, L_2) and (K_0, L_0) cost the same as the input bundle represented by (K_1, L_1). However, (K_1, L_1) is preferred over both (K_2, L_2) and (K_0, L_0) because more quantity is produced by (K_1, L_1) than by (K_2, L_2) or (K_0, L_0). Furthermore, since (K_1, L_1) represents a point of tangency, no other input bundle exists that produces more output than (K_1, L_1).

All that remains to be done is to identify the point of tangency, which occurs when the slopes of the isocost curve and isoquant I_2 are equal. We know from Figure 3.5 and equation (4.5) in the textbook that the slope of the isocost line is $-P_L / P_K$. By definition, the slope of the isoquant is equal to *dK/dL*, since *K* represents the *y*-axis variable in this graph and *L* represents the *x*-axis variable. Equation (4.3) in the textbook (cf. p. 106) implies that $-dK / dL = MP_L / MP_K = MRTS$, where *MRTS* is an acronym for the *marginal rate of technical substitution*.[7] The *MRTS* shows the rate at which one input is substituted for another (holding output constant). Next, we set $P_L / P_K = MP_L / MP_K$, which implies that $MP_L / P_L = MP_K / P_K$. This equation tells us that we should choose an input bundle where the marginal products per dollar spent of labor and capital are the same. If they are not the same, then we should increase the use of the input which has the higher marginal per dollar value.

In closing, I would note that the three class problems which I worked last Thursday, including the Beiswanger Company problem (see p. 113 of the textbook), the Miller Company problem (see p. 114 of the textbook), and the "Wine and Roses" problem, all provide excellent examples concerning how to determine optimal input bundles. I highly recommend that students review these and other similar problems. The concept of returns to scale is also very important, particularly as we move into the next chapter in the textbook on the analysis of costs.

Endnotes

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[1] The law of diminishing returns has been described by Nobel laureate Paul Samuelson and others as one of the most *famous* laws in all of economics; e.g., see Samuelson, P. A. and W. D. Nordhaus, *Microeconomics*, 17th ed. (McGraw Hill 2001), p. 110. Samuelson's contemporary and fellow Nobel laureate, the late Milton Friedman, famously popularized the adage "There's no such thing as a free lunch". This is an important corollary of the law of diminishing returns which communicates (rather effectively, in my opinion), the notion that it is impossible to get something for nothing!

[2] See equation (4.10) on page 117 in the textbook. There, $Q = aL^bK^c$. Set a = 1 and b = c = .5. Then $Q = L^{.5}K^{.5}$.

[3] Note that the marginal product of labor *declines* as we add more of the labor input; i.e., $\partial MP_L / \partial L = -.25L^{-1.5}K^{.5} < 0$. Similarly, the marginal product of capital (MP_k) indicates the rate at which output increases as we add an additional unit of capital, while holding the number of labor inputs constant. Like MP_L, MP_K is positive (i.e., $MP_K = \partial Q / \partial K = .5L^{.5}K^{-.5} > 0$), and if we increase K while holding L constant, then Q increases at a *decreasing* rate; i.e., the marginal product of capital declines as more of the capital input is added (i.e., $\partial MP_K / \partial K = -.25L^{.5}K^{-1.5} < 0$).

[4] For example, $MP_L\big|_{L=1} - MP_L\big|_{L=2} = 1.12 - 0.79 = 0.33 > 0.79 - 0.65 = 0.14 = MP_L\big|_{L=2} - MP_L\big|_{L=3}$, and so forth. The smallest change in the marginal product of labor occurs when the number of labor inputs goes from 9 to 10; i.e., $MP_L\big|_{L=9} - MP_L\big|_{L=10} = 0.37 - 0.35 = 0.02$.

[5] The math is shown here. The output elasticity of labor is η_L = $(L/Q)(\partial Q/\partial L)$ = $(LL^{-.5}K^{-.5})(.5L^{-.5}K^{.5})$ = $.5L^{.5}K^{-.5}K^{.5}L^{-.5} = .5$. Similarly, the output elasticity of capital is η_K = $(K/Q)(\partial Q/\partial K)$ = $(KL^{-.5}K^{-.5})(.5L^{.5}K^{-.5})$ = $.5L^{.5}K^{-.5}K^{.5}L^{-.5} = .5$.

[6] The term "isocost" literally means "same cost"; it implies that all input bundles that fall on this line have the same cost, which is the budgeted amount defined as M. The y-intercept for the isocost line is M / P_K, and its slope is $-P_L / P_K$.

[7] The fact that $MRTS = MP_L / MP_K$ helps to explain why the isoquant initially has a steep slope that diminishes as more labor is substituted in place of capital. The K_2, L_2 input bundle in Figure 3.7 has a very high *MRTS* because MP_L is high and MP_K is low at that point. The K_1, L_1 input bundle has a lower *MRTS* because MP_L is lower at this point than it is at K_2, L_2, and MP_K is higher. This effect becomes even more pronounced at K_0, L_0, which helps explain why the slope of the isoquant at this point is even more shallow.